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Influence of Frequency Noise on Nascent Hysteresis in Optical Bistability

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We study the influence of frequency noise on optical bistability in the neighborhood of the critical point where the hysteresis loop appears. We show that when the transmitted field evolves on a faster time scale than that of the noise, the hysteresis loop shifts toward lower values of the incident pumping field.

KEY WORDS: Frequency noise; hysteresis; optical bistability.

1. INTRODUCTION

The theory of fluctuations in nonequilibrium systems has rapidly grown into a widely diversified area of research. Its development over the last 15 years has been considerably stimulated by the fundamental work of Prigogine on chemical instabilities and dissipative strutures.[1-3] From the beginning, it was evident that noise plays an important role in the onset of these phenomena. Near the bifurcation points, where different branches of solutions of the deterministic equations coalesce, the dynamics of nonequilibrium systems necessarilly involves stochastic elements needing a finer description than the deterministic one. In this domain, the efforts of Prigogine and his co-workers have largely aided to elucidate the analogies existing between phase transitions and nonequilibrium instability (see Ref. 4 for a review).

Recently these analogies have been extended even further. It has been shown that the notion of phase transition applies to transition phenomena due to the presence of external noise. These so-called noise-induced transitions take place when some systemic control parameters randomly fluctuate in the course of time.[5] The source of these fluctuations is in the environment. Their intensity, in contrast to thermal noise, does not scale as

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an inverse power of the system size. This allows deep quantitative and qualitative modifications of the usual bifurcation diagrams. Notably, in some cases one observes that a preexisting, i.e., deterministic, instability is shifted: the noise displaces, by an amount proportional to its intensity, the location in parameter space of an instability already present under noiseless conditions; in other cases, when the intensity of the external noise exceeds some finite threshold, "new" transitions appear, of which there is no trace in the deterministic situation.

Such behavior shows that the role of randomness needs to be appraised for a much broader class of situations than those commonly investigated in which the environment is generally assumed to be constant in time. They call for more systematic studies into the effect of external noise. The results we report here concerning the appearance of hysteresis are a step in this direction in the case of optical bistability.

It is well known that in this phenomenon quantum fluctuations are essential at a fundamental level, and many papers deal with them (see, e.g., Refs. 6–8). Often in experiments, however, the presence of quantum fluctuations is overshadowed by more standard types of noise, such as thermal noise in the material and in the radiation field or by the parametric (external) noise affecting some of the control parameters. The practical importance of knowing how these noises affect optical bistability has given rise to several recent studies (see Ref. 9 and the references cited therein). In particular, in Ref. 10 it has been shown that the presence of frequency noise, which is of multiplicative type, tends to suppress dispersive optical bistability.[3] The treatment used to establish this result is based on a straightforward elimination of the field variables and on the modeling of the frequency fluctuations by Gaussian white noise. Here we revisit the problem under different conditions as far as the noise and the field variables are concerned. Namely, we present a perturbative approach that applies in the neighborhood of the critical point and does not suppose *a priori* that the rate constants describing the evolution of the field variables and of the nonlinear refractive index of the material are widely separated. Furthermore, we consider that the fluctuations of the frequency are given by a colored noise of the Ornstein–Uhlenbeck type.

The system and its deterministic properties are recalled in the next section. In Section 3 we introduce our stochastic treatment and indicate how it can be used to analyze different situations arising when frequency fluctuations are present. In Section 4 we report the results obtained using a perturbative approach in the case where the noise is slow compared to the field variables.

[3] For the standard noise levels this suppression occurs on a time scale which is too long to be relevant for practical purposes, but the phenomenon becomes dramatic for large noise levels.

2. THE MODEL

We consider a purely dispersive bistable device, which under deterministic conditions obeys the evolution equations

$$\dot{x}_1 = -k[x_1 - y + x_2(\Delta N - \theta)] \tag{2.1}$$

$$\dot{x}_2 = -k[x_2 - x_1(\Delta N - \theta)] \tag{2.2}$$

$$(\Delta \dot{N}) = -l(\Delta N - x_1^2 - x_2^2) \tag{2.3}$$

where x_1 and x_2 are the real and imaginary parts of the normalized transmitted field, and y is the normalized incident field, which is taken real and positive for definiteness. ΔN is proportional to the nonlinear part of the refractive index, and θ is the detuning parameter. k and l are the time constants characterizing the intrinsic time scales on which the electric field and the material vary.

The stationary-state solutions x_{1s}, x_{2s}, and $(\Delta N)_s$ of (2.1)–(2.3) obey the well-known cubic steady-state equation[14]

$$E_I^2 \equiv y^2 = |E_s|^2 [1 + (|E_s|^2 - \theta)^2] \tag{2.4}$$

with

$$|E_s|^2 \equiv x_{1A}^2 + x_{2A}^2 = (\Delta N)_1 \tag{2.5}$$

For $\theta \leqslant \theta_c = \sqrt{3}$, the steady-state curve $|E_s|^2$ as a function of y^2 is single-valued (Fig. 1), whereas for $\theta > \sqrt{3}$ it is S-shaped and leads to a hysteresis

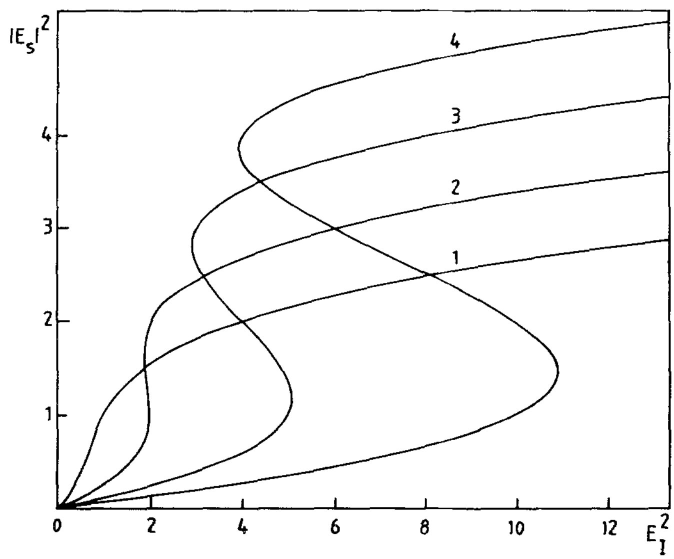

Fig. 1. Steady-state curves of transmitted versus incident intensity for the values of θ indicated.

cycle. The values of the variables and parameters corresponding to the critical point are

$$\begin{gathered} x_{1c} = 3^{1/4}/2^{1/2}, \qquad x_{2c} = -2^{-1/2}3^{-1/2}, \qquad (\Delta N)_c = 2/3^{1/2} \\ y_c = 2 \cdot 2^{1/2}/3^{3/4}, \qquad \theta_c = 3^{1/2} \end{gathered} \tag{2.6}$$

In the following, we study the influence of the fluctuations of the detuning parameter θ in the neighborhood of this critical point. For this purpose it is appropriate to introduce the excess variables and parameters with respect to (2.6) by putting

$$\begin{gathered} x_1 = x_{1c}(1 + \varepsilon u), \qquad x_2 = x_{2c}(1 + \varepsilon v), \qquad \Delta N = (\Delta N)_c(1 + \varepsilon w) \\ \theta = \theta_c(1 + \varepsilon^2 I), \qquad y = y_c(1 + 3\varepsilon^2 I/4 + \varepsilon^3 a) \end{gathered} \tag{2.7}$$

ε is a smallness parameter measuring the distance with respect to criticality; I and a are $O(1)$. Replacing (2.7) into (2.1)–(2.3) and expanding them, one obtains the steady-state solutions u_s, v_s, w_s of the evolution equations as

$$u_s = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$$

$$v_s = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots$$

$$w_s = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots$$

It is easy to see that u_0 is given by the roots of the cubic equation

$$u_0^3 - (3I/4)u_0 - 2a = 0 \tag{2.8}$$

The analogy between (2.8) and the Landau equation for the order parameter in the classical paraferromagnetic phase transition is obvious. Clearly the role of the temperature and external magnetic field are played here respectively by I and a.

3. FLUCTUATIONS OF THE DETUNING PARAMETER θ

We suppose that the detuning parameter θ randomly fluctuates around the constant average value given in (2.7). To incorporate these fluctuations we replacy θ by

$$\theta_t = \theta_c[1 + \varepsilon^2 I + \varepsilon^i(\gamma^{1/2}/Kk^{1/2})^j Z_t] \tag{3.1}$$

where Z_t is an Ornstein–Uhlenbeck process obeying the stochastic differential equation (SDE)

$$dZ_t = -\frac{\gamma}{K^2} Z_t\, dt + \frac{\sigma\gamma^{1/2}}{K} dW_t \tag{3.2}$$

W_t is the Wiener process. The scaling parameter K appearing in (3.1) and (3.2) provides a convenient measure of the "distance" from the white noise situation[11] (cf. also Ref. 4, Chapter 8). This can easily be seen as follows. The transition probability density of Z_t obeys the Fokker–Planck equation (FPE)

$$\partial_t p(Z, t \mid Z_0, 0) = \frac{\gamma}{K^2}\left[\partial_Z Z p(Z, t \mid Z_0, 0) + \frac{\sigma^2}{2}\, \partial_Z p(Z, t \mid Z_0, 0)\right] \tag{3.3}$$

which admits the stationary-state solution

$$p_s(Z) = (\pi\sigma^2)^{-1/2} \exp(-Z^2/\sigma^2) \tag{3.4}$$

The latter is independent of K^2/γ, so that the variance stays constant as the correlation time varies. Assuming that the Ornstein–Uhlenbeck process is started with (3.4), it is a stationary process: one has

$$\langle Z_t \rangle = 0 \tag{3.5}$$

$$C(\tau) = \left\langle \left(\frac{\gamma^{1/2} Z_t}{K}\right)\left(\frac{\gamma^{1/2} Z_{t+\tau}}{K}\right)\right\rangle = \frac{\sigma^2}{2K^2} \exp\left(-\frac{\gamma|\tau|}{K^2}\right) \tag{3.6}$$

Accordingly, the correlation function $C(\tau)$ is exponentially decreasing and the correlation time of the noise is given by

$$\tau_c = K^2/\gamma \tag{3.7}$$

The power spectrum obtained by Fourier transforming (3.6) reads

$$S(\nu) = \frac{\sigma^2}{2\pi(K^4\nu^2/\gamma^2 + 1)} \tag{3.8}$$

Hence, for $K \to 0$, the noise term $\gamma^{1/2} Z_t/K$ that models the fluctuations of θ in (3.1) converges to white noise: its power spectrum becomes flat and at all frequencies $S(\nu) = \sigma^2/2\pi$.

These results also clarify the meaning of the factors ε^i and

$$\frac{1}{\eta} \equiv \left(\frac{\gamma^{1/2}}{K\gamma^{1/2}}\right) = \left(\frac{\tau_f}{\tau_c}\right)^{1/2} \tag{3.9}$$

multiplying Z_t in (3.1). The value of η expresses how rapidely the field variables relax (cf. τ_f) as compared to the fluctuations of θ (cf. τ_c). For $1/\eta \ll 1$ the evolution of the field is much more rapid than that of the noise. ε^i modulates the intensity of the noise inside the system; $i > 0$ corresponds to a weak noise limit.

We set

$$\tau = kt, \qquad \delta = l/k \tag{3.10}$$

replace θ by (3.1) and, using (2.7), we rewrite the dynamical equations of the system in term of the excess variables u, v, w, and Z_τ. One has

$$\frac{du}{d\tau} = -u - \frac{v}{3} + \frac{2w}{3} + \varepsilon\left(\frac{2vw}{3}\right) + \varepsilon^2\left(\frac{4a}{3} - Iv\right) - Z_\tau \frac{\varepsilon^{i-1}}{\eta^j}(1 + \varepsilon v) \tag{3.11}$$

$$\frac{dv}{d\tau} = u - v - 2w + \varepsilon(3I - 2uw) + \varepsilon^2(3Iu) + 3Z_\tau \frac{\varepsilon^{i-1}}{\eta^j}(1 + \varepsilon u) \tag{3.12}$$

$$\frac{dw}{d\tau} = \delta\left[\frac{3u}{2} + \frac{v}{2} - w + \varepsilon\left(\frac{3u^2}{2} + \frac{v^2}{4}\right)\right] \tag{3.13}$$

$$dZ_\tau = -\frac{Z_\tau}{\eta^2}\,d\tau + \frac{\sigma}{\eta}\,dW_\tau \tag{3.14}$$

In the absence of noise, due to the existence of the critical point (2.6), one necessarily has that one of the eigenvalues, say λ_0, associated with the linear terms of (3.11)–(3.13), i.e. the terms of order ε^0, is null. Considering the case where $\delta < 2(1 + \sqrt{2}/3)$, one has that the other two eigenvalues

$$\lambda_\pm = -1 - \tfrac{1}{2}\delta \pm i(\tfrac{1}{3} + \delta - \tfrac{1}{4}\delta^2)^{1/2} = \lambda_r + i\lambda_i \tag{3.15}$$

are then complex conjugate. In order to exploit the critical slowing down due to the zero eigenvalue, we make a change of variables to diagonalize the linear terms of (3.11)–(3.13). We put

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = T \begin{Bmatrix} \varepsilon m \\ \varepsilon m^* \\ n \end{Bmatrix} \tag{3.16}$$

where

$$T = \begin{Bmatrix} -2/(3\delta) & -2(3\delta) & 1 \\ 1 + 2i\lambda_i/\delta & 1 - 2i\lambda_i/\delta & -1 \\ 1 & 1 & 1 \end{Bmatrix}$$

is the transformation matrix formed with the eigenvectors corresponding to λ_0 and $\lambda_\pm$. Furthermore, for simplicity we take $\delta \ll 1$.[4] This situation is typical of miniaturized, optically bistable devices utilizing semiconductors.[12] Decomposing m and m^* into their real and imaginary parts, $m = x + iy$, we obtain for the evolution of x, y, and n a new set of dynamical equations in which we have neglected all higher order terms in δ wherever possible. It is of the form

$$\begin{aligned}
\frac{dx}{d\tau} &= A_0(x, y, n; \delta) + \varepsilon A_1(x, y, n; \delta) + \varepsilon^2 A_2(x, y, n, \delta) \\
&\quad + \frac{\varepsilon^{i-2}}{\eta^j} Z_\tau[A_3(\delta) + \varepsilon A_4(n, \delta) + A_5(x, y; \delta) \\
&\equiv X(x, y, n, z; \varepsilon, \delta) \qquad (3.17)
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{d\tau} &= B_0(x, y, n; \delta) + \varepsilon B_1(x, y, n; \delta) + \varepsilon^2 B_2(x, y, n; \delta) \\
&\quad + \frac{\varepsilon^{i-2}}{\eta^j} Z_\tau[B_3(\delta) + \varepsilon B_4(n, \delta) + B_5(x, y; \delta) \\
&= Y(x, y, n, z; \varepsilon, \delta) \qquad (3.18)
\end{aligned}$$

$$\begin{aligned}
\frac{dn}{d\tau} &= \varepsilon^2 C_2(x, y, n; \delta) + \varepsilon^3 C_3(x, y, n; \delta) \\
&\quad + \frac{\varepsilon^{i-1}}{\eta^j} Z_\tau[C_4(\delta) + \varepsilon C_5(n; \delta) + \varepsilon^2 C_6(n, \delta) \\
&\equiv F(x, y, n, z; \varepsilon, \delta) \qquad (3.19)
\end{aligned}$$

The joint probability density $p(x, y, n, z, t)$ of the system obeys the following FPE:

$$\begin{aligned}
&\partial_t p(x, y, n, z, t) \\
&\quad = \Bigg[-\partial_x X(x, y, n, z; \varepsilon, \delta) - \partial_y Y(x, y, n, z; \varepsilon, \delta) \\
&\qquad - \partial_n F(x, y, n, z; \varepsilon, \delta) + \frac{1}{\eta^2}\left(\partial_z z + \frac{\sigma^2}{2}\partial_{zz}\right)\Bigg] p(x, y, n, z, t) \qquad (3.20)
\end{aligned}$$

In the following we suppose that the system has been coupled with the stationary noise Z_τ for a sufficiently long time so that it has had the

[4] This simplification avoids the handling of very lengthy polynomial expressions. It is not the basis of the adiabatic elimination procedure.

possibility to settle down to a steady state: $\partial_t p = 0$. We are interested in the corresponding probability density $p_s(x, y, n, z)$. Its evaluation in general is an intractable mathematical problem. For some physically relevant limiting cases, however, an approximate solution can be obtained by perturbation methods.

Two situations can in fact be treated straightforwardly starting from (3.20). The first arises when

$$\varepsilon \ll 1 \ll 1/\eta \tag{3.21}$$

i.e., the field evolves much more slowly than the noise, but both these variables evolve much more rapidly than n. It is then appropriate to modulate the intensity of the noise in (3.1) by putting $j = 1$ and $i = 2$. Indeed, for $\eta^{-1} \to \infty$, the correlation function of Z_τ/η,

$$\langle (Z_\tau/\eta)(Z_0/\eta) \rangle = (\sigma^2/2\eta^2) \exp(-|T|/\eta^2) \tag{3.22}$$

becomes δ-correlated, indicating that the corresponding power spectrum becomes white; $p_s(x, y, n, z)$ in this case can be evaluated by a technically lengthy procedure which has already been used and described in detail in the case of the Hopf bifurcation.[13] We hope to come back to this problem in a forthcoming paper. In the next section, we devote our attention to the other case, for which (3.20) is the appropriate starting point, namely that $\eta^{-1} \ll 1$, i.e., the field variations are more rapid than the fluctuations of the noise.

4. SOLUTION OF (3.20) WHEN THE FIELD EVOLVES MORE RAPIDELY THAN THE NOISE

It is then appropriate to modulate the intensity of the noise by putting $i = 2$ and $j = 0$. Explicitly the drift terms for x, y, and n in (3.20) then read

$$\begin{aligned}
&X(x, y, n, z; \varepsilon, \delta) \\
&\quad = -x - \frac{y}{\sqrt{3}} + \frac{\delta n^2}{2} + Z_\tau\left(\frac{3\delta}{2}\right) + \varepsilon\left[-\frac{3\delta}{2} xn + \frac{2}{\sqrt{3}} yn - \delta a\right. \\
&\quad\quad \left. - \frac{3\delta I}{4} n + Z_\tau\left(-\frac{3\delta n}{4}\right)\right] + \varepsilon^2\left[x^2 + y^2 + \frac{4xy}{\sqrt{3}} + \frac{3\delta I}{2} x\right. \\
&\quad\quad \left. - \sqrt{3} I y + Z_\tau\left(\frac{3\delta x}{2} - \sqrt{3} y\right)\right] \\
&\quad = h_0 + \varepsilon h_1 + \varepsilon^2 h_2
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&Y(x, y, n, z; \varepsilon, \delta)\\
&= \frac{x}{\sqrt{3}} - y + \frac{\sqrt{3}\,\delta}{2} n^2 - \frac{3\sqrt{3}\,\delta I}{4} - Z_\tau \left(\frac{3\sqrt{3}\,\delta}{4}\right) + \varepsilon \left[-\frac{2}{\sqrt{3}} xn + \frac{3\delta}{2} yn \right.\\
&\quad \left. - \sqrt{3}\,\delta^2 a - \frac{3\sqrt{3}\,\delta I}{4} n - Z_\tau \left(\frac{3\sqrt{3}\,\delta n}{4}\right)\right] + \varepsilon^2 [-5\sqrt{3}\,x^2 - \sqrt{3}\,y^2\\
&\quad + 5\delta xy + \sqrt{3}\,Ix - 3\delta Iy + Z_\tau(\sqrt{3}\,x - 3\delta y)]\\
&= g_0 + \varepsilon g_1 + \varepsilon^2 g_2
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&F(x, y, n, z; \varepsilon, \delta)\\
&= -\frac{3\varepsilon\delta}{2} Z_\tau + \varepsilon^2 \left[-2xn - \frac{2}{\sqrt{3}} yn + 2\delta a + \frac{3\delta I}{2} n \right.\\
&\quad \left. + Z_\tau \left(\frac{3\delta}{2} n\right)\right] + \varepsilon^3 \left[\frac{4}{3\delta}(x^2 + y^2) - 4\sqrt{3}\,xy - 3\delta Ix \right.\\
&\quad \left. + 2\sqrt{3}\,Iy + Z_\tau \left(2\sqrt{3}\,y - \frac{2\delta}{2} x\right)\right]\\
&= \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3
\end{aligned} \tag{4.3}$$

We replace (4.1)–(4.3) in (3.20), and assuming that

$$\varepsilon \ll 1/\eta^2 \ll 1, \qquad \varepsilon^3/\delta \ll 1/\eta^2 \tag{4.4}$$

we expand $p_s(x, y, n, z)$ as

$$p_s(x, y, n, z) = p_0(x, y, n, z) + \eta^{-2} p_2(x, y, n, z) + \cdots \tag{4.5}$$

Replacing (4.5) in (3.20) with $\partial_t p = 0$, we find at the order η^0 that

$$(\partial_x h_0 + \partial_y g_0)\, p_0(x, y, n, z) = 0 \tag{4.6}$$

From (6) one immediately deduces that

$$p_0(x, y, n, z) = \delta(x - x_0)\, \delta(y - y_0)\, \phi(n, z) \tag{4.7}$$

where

$$x_0 = \frac{9\delta}{8}\left(\frac{I}{2} + Z_\tau\right) \tag{4.8}$$

$$y_0 = \frac{\sqrt{3}\,\delta}{2}\left(n^2 - \frac{9I}{8} - \frac{3Z_\tau}{4}\right) \tag{4.9}$$

are the solutions of $h_0(x_0, y_0, n, z) = 0$ and $g_0(x_0, y_0, n, z) = 0$, and where $\phi(n, z)$ remains a function to be determined. This is what one expects to happen when the evolution of the field variables is so fast that at each instant they are able to "equilibrate" to the value of the noise. Another way to see this is to consider the correlation function

$$\langle Z_\tau Z_0 \rangle = \tfrac{1}{2}\sigma^2 \exp(-|\tau|/\eta^2) \tag{4.10}$$

One notes that η^2 is nothing else than the correlation time of the noise when the unit of time is taken equal to the relaxation time of the field. Hence, in the limit $\eta^{-2} \to 0$, which is of interest here, the evolution of the field is completely correlated to that of Z_τ: the spectral density $S(\nu)$ of Z_τ converges to a δ-peak located at the frequency $\nu = 0$, i.e.,

$$\lim_{\eta^{-2} \to 0} S(\nu) = \tfrac{1}{2}\sigma^2 \delta(\nu) \tag{4.11}$$

In order to determine $\phi(n, z)$, one notes that at the order η^{-2} the following solvability condition must be imposed:

$$\int_{\mathbb{R}} dx\, dy \left[-\partial_n(\varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3) + \partial_z z + \frac{\sigma^2}{2}\partial_{zz} \right]$$
$$\times\, \delta(x - x_0)\, \delta(y - y_0)\, \phi(n, z) = 0$$

This yields for $\phi(n, z)$ the equation

$$\left(\partial_z z + \frac{\sigma^2}{2}\partial_{zz} \right) \phi(n, z) = \partial_n(\varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3)\, \phi(n, z) \tag{4.12}$$

Expanding $\phi(n, z)$ as

$$\phi(n, z) = \tilde{\phi}_0(n, z) + \varepsilon \tilde{\phi}_1(n, z) + \cdots \tag{4.13}$$

one finds at the order ε^0 that

$$\tilde{\phi}_0(n, z) = p_s(z)\, \phi_0(n) \tag{4.14}$$

i.e., the joint probability factorizes. This is natural since n undergoes a critical slowing down. Setting now $\tilde{\phi}_k(n, z) = p_s(z)\, \phi_k(n, z)$ $(k = 1,\ldots)$, one has at the order ε^1 for $\phi_1(n, z)$ the equation

$$(-z\partial_z + \tfrac{1}{2}\sigma^2 \partial_{zz})\, \phi_1(n, z) = \partial_n f_1 \phi_0(n) \tag{4.15}$$

Again a solvability condition, namely

$$\int_{\mathbb{R}} dz\, \partial_n[f_1\, p_s(z)\, \phi_0(n)] = 0 \tag{4.16}$$

must be imposed. Given that $f_1 = -3\delta z/2$ and that $p_s(z)$ is a Gaussian, (4.16) is automatically satisfied. The solution of (4.16) can then be written as

$$\phi_1(n, z) = H_1(n) - z\partial_n(-\tfrac{3}{2}\delta)\,\phi_0(n) \tag{4.17}$$

where $H_1(n)$ remains to be determined. Proceeding now to the order ε^2, one has that

$$(-z\partial_z + \tfrac{1}{2}\sigma^2\partial_{zz})\,\phi_2(n, z) = \partial_n f_1 \phi_1(n, z) + \partial_n f_2 \phi_0(n) \tag{4.18}$$

The solvability condition applied to the right-hand side of (4.18) yields the equation

$$-\frac{\sigma^2}{2}\left(\frac{9\delta^2}{4}\right)\partial_{nn}\phi_0(n) + \partial_n\left(-\delta n^3 + \frac{3\delta I}{2} n + 2\delta a\right)\phi_0(n) = 0$$

whose solution

$$\phi_0(n) = N \exp\left[\frac{2}{9\sigma^2\delta}(-n^4 + 3In^2 + 8an)\right] \tag{4.19}$$

determines $p_0(x, y, n, z)$ completely:

$$p_0(x, y, n, z) = N\,\delta(x - x_0)\,\delta(y - y_0)\,p_s(z)\exp\left[\frac{2}{9\sigma^2\delta}\,(-n^4 + 3In^2 + 8an)\right] \tag{4.20}$$

N is a normalization constant. In order to determine the solution completely up to the order ε^1, the solvability conditions must be calculated up to the order ε^3. Evaluating the solution of (4.8), one obtains

$$\phi_2(n, z) = H_2(n) + z^2\left[\frac{9\delta^2}{8}\,\partial_{nn}\phi_0(n)\right] + z\left[\frac{3\delta}{2}\,\partial_n H_1(n)\right] \tag{4.21}$$

Using (4.21) and the expressions for $\phi_1(n, z)$ and $\phi_0(n)$, one deduces from the solvability condition at the order ε^3 the condition

$$-\frac{\sigma^2}{2}\left(\frac{9\delta^2}{4}\right)\partial_{nn}H_1(n) + \partial_n f_2 H_1(n)$$
$$+ \partial_n\left[\frac{\sigma^2}{2}\left(\frac{27\delta^2}{8}\right) + \delta n^4 + \frac{3\delta I}{4}n^2 + \frac{135\delta I^2}{64}\right]\phi_0(n) = 0 \tag{4.22}$$

Putting $H_1(n) = \tilde{H}_1(n)\,\phi_0(n)$ and solving for $\tilde{H}_1(n)$ yields

$$\tilde{H}_1(n) = \frac{2}{\sigma^2}\left(\frac{4}{9\delta^2}\right)\left\{-C_1 \int^n \frac{du}{\phi_0(u)} + \int^n L(u)\,du + C_2\right\}$$

with

$$L(u) = \frac{\sigma^2}{2}\left(\frac{27\delta^2}{8}\right) + \delta\left(u^4 + \frac{3Iu^2}{4} + \frac{135I^2}{64}\right) \tag{4.23}$$

and where C_1 and C_2 are constants. We expect the moments of the distribution to exist, i.e.,

$$\int_{\mathbb{R}} dz \int_{\mathbb{R}} dn\, n^k \phi(n, z) < \infty \tag{4.24}$$

Hence we have the condition

$$\int_{\mathbb{R}} n^k \tilde{H}_1(n)\,\phi_0(n)\,dn < \infty$$

which for $k \geqslant 2$ can only be satisfied if $C_1 = 0$. Next the normalisation condition

$$\int dx\,dy\,dn\,dz\; p_k(x, y, n, z) = 0, \qquad k = 1, 2,...$$

yields the relation

$$\int_{\mathbb{R}} \left[\int^n L(u)\,du + C_2\right] \phi_0(n)\,dn = 0 \tag{4.25}$$

which determines the value of C_2. Thus, up to the order ε^1, we obtain finally

$$p_s(x, y, n, z)$$
$$= N\delta(x - x_0)\,\delta(y - y_0)\left\{1 + \varepsilon\left[C_2 + F(n) + z\left(\frac{3\delta}{2}\right)\partial_n\right]\right\}\phi_0(n)\,p_s(z) \tag{4.26}$$

where

$$F(n) = \frac{8}{9\sigma^2\delta}\left[\frac{n^5}{5} + \frac{In^3}{4} + \delta\left(\frac{135I^2}{64} + \frac{27\sigma^2}{16}\right)n\right]$$

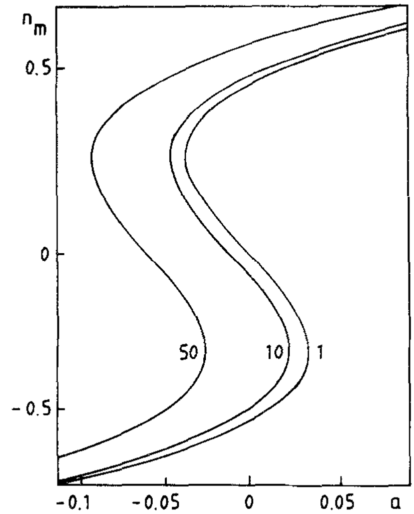

Fig. 2. Localization of the extreme n_m as a function of a for the values of σ^2 indicated and $I = 0.2$, $\delta = 0.1$, $\varepsilon = 0.05$.

From (4.26) we can immediately determine the reduced probability density $p_s^*(n)$ for n:

$$p_s^*(n) = \int dz\, dx\, dy\; p_s(x, y, n, z)$$

$$= \{1 + \varepsilon[C_2 + F(n)]\}\, \phi_0(n) \tag{4.27}$$

In Fig. 2, we have plotted its extrema u_m as a function of a for increasing values of σ^2 and a noiseless situation chosen slightly inside the bistable domain. One observes that the frequency noise shifts the bistable region to smaller values of the incident field.

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